increased flow velocity, leading to increased $\dot{M}$ and $\dot{Q}$. For a practising engineer the maximum temperature along the left and right interfaces is useful information, given in non-dimensional form in Tables 3 and 4. In general, the impact of wall conduction on $\dot{M}$ and $\dot{Q}$ decreases with decrease in γ_T.

CONCLUSIONS

In the numerical study of the effect of wall conduction on laminar heat transfer between two vertical plates subjected to asymmetric heating, walls are heated by subjecting their external surface to constant temperature. The governing equations were solved by an implicit finite difference technique. Calculations were made for a wide range of independent parameters (Gr, t/B, K, L/B and γ_T). The heat transfer and fluid flow in the channel are proportional to the buoyancy forces. Higher values of Gr, K and γ_T contribute to higher buoyancy forces; lower values of t/B result in higher buoyancy forces. The quantitative effect of wall conduction on $\dot{M}$ and $\dot{Q}$ under asymmetric heating conditions can be summarized as follows.

(1) For $Gr = 10^4$, $K = 1$, $t/B = 0.5$, $L/B = 1$ and $\gamma_T = 1$, $\dot{M}$ decreases to 51.6% of the $\dot{M}$ for $t/B = 0$. The heat flow rate reduces to 23.1% of the $\dot{Q}$ for $t/B = 0$. This indicates that the wall conduction reduces $\dot{M}$ and $\dot{Q}$ for $\gamma_T = 1$.

(2) For $Gr = 10^4$, $K = 1$, $t/B = 0.5$, $L/B = 1$ and $\gamma_T = 0$, $\dot{M}$ reduces to 66.7% of the $\dot{M}$ for $t/B = 0$. The $\dot{Q}$ reduces to 26.2% of $\dot{Q}$ for $t/B = 0$. This implies that the asymmetric heating ($\gamma_T = 0$) has less impact on $\dot{M}$ and $\dot{Q}$ than the wall conduction.

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The effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous medium

F. C. Lai and F. A. Kulacki†

Department of Mechanical Engineering, Colorado State University, Fort Collins, CO 80523, U.S.A.

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INTRODUCTION

In most of the previous studies on heat transfer in saturated porous media, the thermophysical properties of fluid were assumed to be constant. However, it is known that these properties may change with temperature, especially for fluid viscosity. To accurately predict the heat transfer rate, it is necessary to take into account this variation of viscosity. In spite of its importance in many applications, this effect has received rather little attention.

Previous results [1–4] have shown that when the effects of variable viscosity are taken into consideration, the critical Rayleigh number for the onset of convection is substantially reduced from the classical value, although the associated wave number is nearly the same. For a two-dimensional cavity, it is found that the flow and temperature fields become unstable at even moderate values of the Rayleigh number and exhibit a fluctuating convective state analogous to that observed for the constant viscosity case. In summary, previous studies have considered mostly the instability of the flow and temperature fields caused by the variation of viscosity. Heat transfer results, however, are still very limited. For reported heat transfer data, the working fluids considered are mainly liquids, especially water. For gases, viscosities vary quite differently from liquids. Therefore, it is expected that the heat transfer results for gases also be different from those in liquids. This discrepancy in heat transfer will be further elaborated upon in the following analysis.

In this note, the effect of variable viscosity is considered for mixed convection along a vertical plate embedded in a saturated porous medium. The limiting cases of natural and forced convection are also examined. Similarity solutions are obtained for an isothermally heated plate with fluid viscosity varied as an inverse function of temperature. As pointed out by Cheng and Minkowycz [5], problems of this kind have important applications in geophysics, particularly, geothermal energy extraction and underground storage systems. In addition, it also finds very useful applications in the design of insulation systems employing porous media.

ANALYSIS

Consider a vertical plate embedded in a saturated porous medium. The fluid and medium properties are assumed to be isotropic and constant, except for the fluid viscosity. The governing equations based on Darcy's law are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u = -\frac{K}{\mu}\left(\frac{\partial p}{\partial x} + \rho g\right) \tag{2}$$

† Author to whom all correspondence should be addressed.

NOMENCLATURE

a constant defined in equation (9a) $[\mathrm{K^{-1}\,Pa^{-1}\,s^{-1}}]$
f dimensionless stream function
g acceleration due to gravity $[\mathrm{m\,s^{-2}}]$
h local heat transfer coefficient $[\mathrm{W\,m^{-2}\,K^{-1}}]$
K permeability $[\mathrm{m^2}]$
k effective thermal conductivity of the saturated porous medium $[\mathrm{W\,m^{-1}\,K^{-1}}]$
Nu local Nusselt number, hx/k
Pe Peclet number, $U_\infty x/\alpha$
p pressure [Pa]
Ra Rayleigh number, $Kg\beta(T_0-T_\infty)x/\nu_\infty\alpha$
T temperature [K]
T_e constant defined in equation (9b) [K]
T_0 temperature of plate [K]
T_∞ ambient temperature [K]
u, v velocity components in x- and y-direction $[\mathrm{m\,s^{-1}}]$
U_∞ free stream velocity $[\mathrm{m\,s^{-1}}]$
x, y Cartesian coordinates [m].

Greek symbols
α effective thermal diffusivity $[\mathrm{m^2\,s^{-1}}]$
β coefficient of thermal expansion $[\mathrm{K^{-1}}]$
γ constant defined in equation (5) $[\mathrm{K^{-1}}]$
δ_T thermal boundary-layer thickness [m]
η independent similarity variable
η_T dimensionless thermal boundary-layer thickness
θ dimensionless temperature
θ_e constant defined in equation (19)
μ dynamic viscosity [Pa s]
ν kinematic viscosity $[\mathrm{m^2\,s^{-1}}]$
ρ fluid density $[\mathrm{kg\,m^{-3}}]$
Ψ stream function.

Subscripts
fc forced convection
mx mixed convection
nc natural convection
0 condition at the wall
∞ property related to reference state.

$$v = -\frac{K}{\mu}\frac{\partial p}{\partial y} \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

$$\frac{1}{\mu} = \frac{1}{\mu_\infty}[1 + \gamma(T - T_\infty)] \tag{5}$$

where the viscosity of fluid has been assumed to be an inverse linear function of temperature. This is a reasonably good approximation for liquids such as water and crude oil [6].

The boundary conditions are given by

$$y = 0, \quad T = T_0, \quad v = 0 \tag{6}$$

$$y \to \infty, \quad T = T_\infty, \quad u = 0 \text{ (natural convection)} \tag{7a}$$

$$= U_\infty \text{ (mixed and forced convection).} \tag{7b}$$

Equation (5) can be rewritten as

$$\frac{1}{\mu} = a[T - T_e] \tag{8}$$

where $a = \gamma/\mu_\infty$ and $T_e = T_\infty - 1/\gamma$. Both a and T_e are constant, and their values depend on the reference state and the thermal property of the fluid, i.e. γ. In general, $a > 0$ for liquids, and $a < 0$ for gases. (The viscosity of a liquid usually decreases with increasing temperature while it increases for gases.)

To further demonstrate the appropriateness of equation (5), correlations between viscosity and temperature for air and water are given below because these two are the most common working fluids found in engineering applications.

For air

$$\frac{1}{\mu} = -123.2(T - 742.6), \quad \text{based on } T_\infty = 293 \text{ K } (20°\text{C}) \tag{9}$$

and for water

$$\frac{1}{\mu} = 29.83(T - 258.6), \quad \text{based on } T_\infty = 288 \text{ K } (15°\text{C}). \tag{10}$$

The data used for these correlations are taken from ref. [7]. While equation (9) is good to within 1.2% from 278 K (5°C) to 373 K (100°C), equation (10) is good to within 5.8% from 283 K (10°C) to 373 K (100°C). The reference temperatures thus selected for the correlations are very practical in most applications.

Having invoked the Boussinesq and boundary-layer approximation, the governing equations in terms of stream function, Ψ, are

$$(T - T_e)\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial T}{\partial y}\frac{\partial \Psi}{\partial y} + K\rho_\infty g\beta a(T - T_e)^2\frac{\partial T}{\partial y} \tag{11}$$

$$\frac{\partial \Psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \tag{12}$$

With the properly chosen similarity variables, equations (11) and (12) can be transformed to a set of ordinary differential equations.

Natural convection

The suitable similarity variables for natural convection are

$$\eta = (Ra)^{1/2}y/x \tag{13}$$

$$\Psi = \alpha(Ra)^{1/2}f(\eta) \tag{14}$$

$$\theta = (T - T_\infty)/(T_0 - T_\infty). \tag{15}$$

After transformation, the resulting equations are

$$f'' = \frac{1}{\theta - \theta_e}f'\theta' - \frac{\theta - \theta_e}{\theta_e}\theta' \tag{16}$$

$$\theta'' = -\tfrac{1}{2}f\theta' \tag{17}$$

where θ_e is a constant and is defined by

$$\theta_e = \frac{T_e - T_\infty}{T_0 - T_\infty} = -\frac{1}{\gamma(T_0 - T_\infty)}. \tag{18}$$

Its value is determined by the viscosity of the fluid in consideration and the operating temperature difference. A large value of θ_e implies either γ or $(T_0 - T_\infty)$ are small, and the effects of variable viscosity can thus be neglected. On the other hand, for a smaller value of θ_e, either the fluid viscosity changes markedly with temperature or the operating temperature difference is high. In either case, the variable viscosity effect is expected to become very important. Also bearing in mind that the liquid viscosity varies differently with temperature than that of gas, therefore, it is important to note that θ_e is negative for liquids and positive for gases. The concept of this new parameter θ_e was first introduced by Ling and Dybbs [6] in their study of forced convective flow in porous media. However, they did not recognize the significance of the sign of θ_e. Although they intended to investigate the variable viscosity effect of liquids, the results actually applied to gases.

The boundary conditions are

$$\eta = 0, \quad \theta = 1, \quad f = 0 \tag{19}$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 0. \tag{20}$$

With the aid of boundary condition (20), equation (16) can be integrated once to give

$$f' = -\frac{\theta(\theta-\theta_e)}{\theta_e}. \tag{21}$$

Mixed convection

The appropriate similarity variables for mixed convection are

$$\eta = (Pe)^{1/2}y/x \tag{22}$$

$$\Psi = \alpha(Pe)^{1/2}f(\eta). \tag{23}$$

With the aid of boundary condition (7b), equations (11) and (12) are transformed to

$$f' = -\frac{\theta-\theta_e}{\theta_e}\left(\frac{Ra}{Pe}+1\right) \tag{24}$$

$$\theta'' = -\tfrac{1}{2}f\theta' \tag{25}$$

and the corresponding boundary conditions are

$$\eta = 0, \quad \theta = 1, \quad f = 0 \tag{26}$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 1. \tag{27}$$

Forced convection

It is noted that the governing equations for forced convection can be readily deduced from equations (24) and (26) by setting $Ra/Pe = 0$. Therefore

$$f' = -\frac{\theta-\theta_e}{\theta_e} \tag{28}$$

$$\theta'' = -\tfrac{1}{2}f\theta'. \tag{29}$$

RESULTS AND DISCUSSION

The transformed ordinary differential equations, with the corresponding boundary conditions, are solved by numerical integration using the Runge–Kutta method and by the shooting technique with a systematic guessing of $-\theta'(0)$. To verify the proper treatment of the problem, the solutions have been compared with those of the corresponding constant-viscosity cases [5, 8] by setting $\gamma = 0$. The results are in excellent agreement.

As discussed earlier, a new parameter, θ_e, is introduced to the present analysis to take into account the variation of fluid viscosity. To reveal its influence on the flow and temperature fields, the slip velocity and local heat flux are shown in Figs. 1 and 2, respectively.

It is clearly seen that the slip velocity, i.e. $f'(0)$, is significantly influenced by the value of θ_e. In addition, it is found that the slip velocities for natural convection are identical to those for forced convection. This is verified by substituting $\eta = 0$ to equations (21) and (28). For both cases, the velocity is increased as $\theta_e \to 0$ for $\theta_e < 0$ and decreased as $\theta_e \to 0$ for $\theta_e > 0$. For mixed convection, it also increases with Ra/Pe. Nevertheless, it is noticed that the slip velocity is reduced to zero for $\theta_e = 1$ due to an increase in the fluid viscosity. A

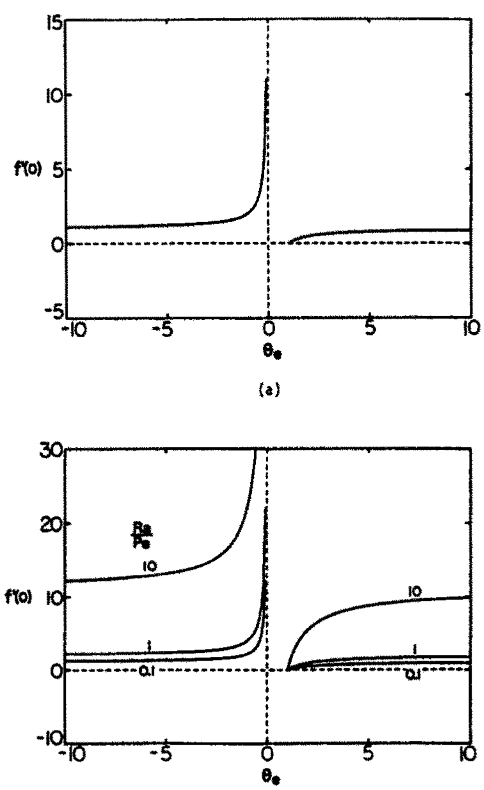

FIG. 1. Variation of slip velocity as a function of θ_e: (a) for free and forced convection; (b) for mixed convection.

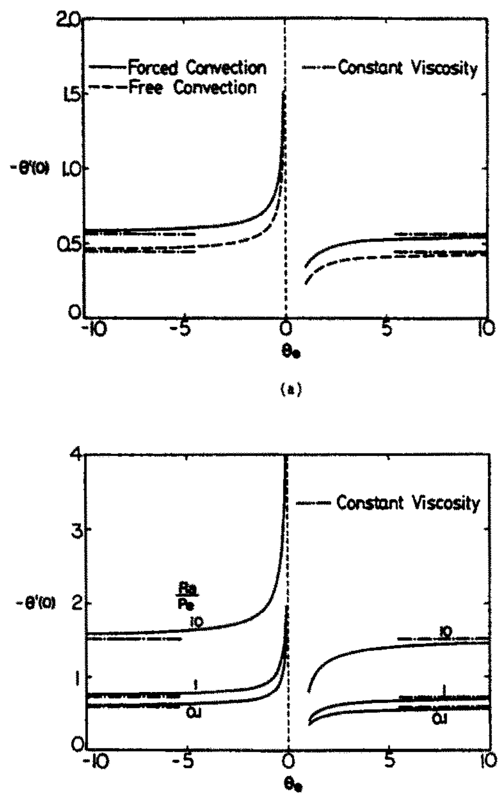

FIG. 2. Variation of local heat flux at wall as a function of θ_e: (a) for free and forced convection; (b) for mixed convection.

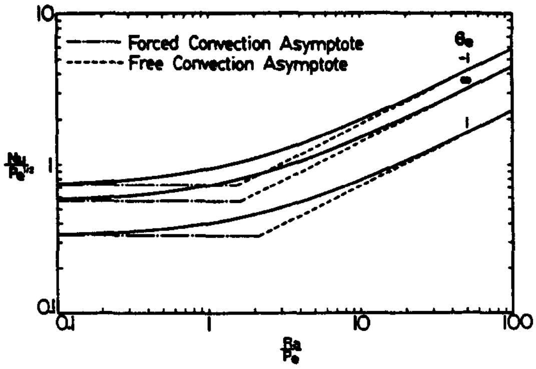

FIG. 3. Heat transfer results as a function of Ra/Pe.

further increase in the fluid viscosity, i.e. $\theta_e < 1$, will result in an adverse pressure gradient and separation of flow will occur. In that case, boundary-layer approximation fails and similarity solutions no longer exist.

It is observed that for $\theta_e < 0$, the solution of local heat flux at the wall is increased considerably as $\theta_e \to 0$ while it is dramatically reduced as $\theta_e \to 1$ for $\theta_e > 0$. In addition, it asymptotically approached that of the constant-viscosity case as $\theta_e \to \pm\infty$, which implies either γ or $(T_0 - T_\infty)$ are very small. In either case, this means the variation of fluid viscosity is negligible.

For applications in geothermal engineering, the practical interests are the thermal boundary-layer thickness and heat transfer rate. Consider thermal boundary-layer thickness first. It is found that the effect of variable fluid viscosity is to thicken the thermal boundary layer for $\theta_e > 0$ as θ_e approaches unity while it is suppressed for $\theta_e < 0$ as θ_e approaches zero.

The heat transfer coefficient in terms of the Nusselt number is expressed as

$$\frac{Nu}{Ra^{1/2}} = -[\theta'(0)]_{nc} \text{ for free convection} \quad (30)$$

and

$$\frac{Nu}{Pe^{1/2}} = -[\theta'(0)]_{mx} \text{ for mixed convection} \quad (31)$$

$$= -[\theta'(0)]_{fc} \text{ for forced convection.} \quad (32)$$

Equation (31) is plotted in Fig. 3 as a function of Ra/Pe. The limiting cases of free and forced convection are also shown as asymptotes in the same figure. The influence of variable viscosity on the heat transfer results can be clearly observed from this figure. For $\theta_e < 0$, the heat transfer rate is greater than that for the constant-viscosity case while it is less for $\theta_e > 0$.

The free convection asymptotes can be obtained by rewriting equation (31) as

$$\frac{Nu}{Pe^{1/2}} = \frac{Nu}{Ra^{1/2}}\left(\frac{Ra}{Pe}\right)^{1/2} = \left(\frac{Ra}{Pe}\right)^{1/2}[-\theta'(0)]_{nc}. \quad (33)$$

For a given Ra, $[-\theta'(0)]_{nc}$ can be obtained by solving simultaneous equations (17) and (21). Once $[-\theta'(0)]_{nc}$ is known, the free convection asymptotes can be constructed from equation (33).

CONCLUSION

For applications in geophysics and insulation engineering, an operating temperature difference of 80 K is fairly common. This gives $\theta_e = 5.62$ for air and $\theta_e = -0.37$ for water. From the results of the present study, it is estimated that, for air, the heat transfer rate based on the constant viscosity assumption is about 6% higher than that of the present results, and 40% lower for water. Therefore, we can conclude that when the viscosity of a working fluid is sensitive to the variation of temperature or the temperature difference of the application is large, the variable viscosity effect has to be taken into consideration. When neglected, it can result in a considerable error in the heat transfer coefficient.

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